Toward classification of the singular fibers of minimal degenerations of type I of surfaces with K=0

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Toward classification of the singular fibers of minimal degenerations of type I of surfaces with $\kappa = 0$

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1 Introduction

In [K.Ohno], we defined *the log minimal reduction* and *the log minimal degeneration*, and determined the singular fibres in the case when the support of the singular fibre of associated log minimal degeneration contains a smooth elliptic curve as a double curve. The minimal degenertion with the above condition may be called the *type II degeneration*.

In this report, we launch into classification of the singular fibres of minimal degenerations of surfaces with $\kappa = 0$, in the case when the support of the singular fibre of associated log minimal degeneration is irreducible. The minimal degeneration with this condition may be called the *type I degeneration*. In fact, this degeneration corresponds to *the first kind degeneration* in the sense of [K.Ueno 71]. Let $\hat{f} : (\hat{X}, \hat{\Theta}) \to \mathcal{D}$ be a log minimal degeneration with $\kappa = 0$ over the complex disk $\mathcal{D}$. i.e., $\hat{X}$ is normal Q-factorial 3-fold, $\hat{\Theta} := \hat{f}^*(0)_{\mathrm{red}}$, $(\hat{X}, \hat{\Theta})$ is strictly log terminal, $\hat{f}$ is projective connected morphism to a complex disk $\mathcal{D}$, $K_{\hat{X}} + \hat{\Theta}$ is $\hat{f}$-nef and $\hat{X}_t := \hat{f}^*(t)$ is smooth surface with $\kappa = 0$ for $t \in \mathcal{D}^* := \mathcal{D} \setminus \{0\}$. Let notations be as above. Then $(\hat{\Theta}, \mathrm{Diff}_{\hat{\Theta}}(0))$ is a ν_0-*log surface of type I* in the following sense.

Definition 1.1 Let (S, Δ) be a normal log surface. (S, Δ) is called ν_0-*log surface of type I*, If the following conditions are satisfied.

(*i*) (S, Δ) is Kawamata log terminal.

(*ii*) $K_S + \Delta \sim_{\mathrm{num}} 0$.

(*iii*) Δ is written as $\Delta = \sum_i \{(m_i - 1)/m_i\}\Delta_i$, where Δ_i is irreducible and $m_i \in \boldsymbol{N}$ for any i.

We note that ν_0-log surface of type I is a Log Enriques surface in the sense of De-Qi Zhang [D.-Q. Zhang 91], If $\Delta = 0$ and $q(S) = 0$.

Definition 1.2 Let (S, Δ) be a ν_0-log surface of type I.

$$\mathrm{CI}(S, \Delta) := \mathrm{Min}\{n \in \boldsymbol{N}; n(K_S + \Delta) \text{ is Cartier}\}$$

is called *the Cartier index of* (S, Δ).

Definition 1.3 Let (S, Δ) be as above.

$$\mathrm{GI}(S, \Delta) := \mathrm{Min}\{n \in \boldsymbol{N}; n(K_S + \Delta) \sim 0\}$$

is called *the global index of* (S, Δ).

Let (S, Δ) be as above and put $r := \mathrm{GI}(S, \Delta)$. We define the *log canonical cover* of (S, Δ) as

$$\pi : \tilde{S} := \mathrm{Spec}_S \oplus_{i=0}^{r-1} \mathcal{O}_S(\lfloor -i(K_S + \Delta) \rfloor) \to S,$$

where $\mathcal{O}_S$-algebra structure of $\oplus_{i=0}^{r-1} \mathcal{O}_S(\lfloor -i(K_S + \Delta) \rfloor)$ is given by a nowhere vanishing section of $\mathcal{O}_S(r(K_S + \Delta))$ and this definition does not depend on the choice of the nowhere vanishing section up to isomorphism. From the definition and [VV.Shokurov 93], Corollary 2.2, $\tilde{S}$ is a normal surface with only rational double points and has trivial canonical bundle. So $\tilde{S}$ is a K3 surface with only rational double points or abelian surface by classification theory of surfaces.

Definition 1.4 Let (S, Δ) be a ν_0-log surface of type I, and $\pi : \tilde{S} \to S$ be the log canonical cover. When $\tilde{S}$ is a K3 surface with only rational double points (resp., smooth K3 surface, resp., abelian surface), (S, Δ) is called *ν_0-log surface of type K3* (resp., *special ν_0-log surface of type K3*, resp., *ν_0-log surface of abelian type*).

The next lemma gives us the hope of classifying ν_0-log surfaces.

Lemma 1.1 ([V.Nikulin 80] Theorem 3.1, [D.-Q. Zhang 91] Lemma 2.3) *If (S, Δ) is a ν_0-log surface of type K3, then $\varphi(CI(S, \Delta))|22 - \tilde{\rho}$. If (S, Δ) is a ν_0-log surface of abelian type and $GI(S, \Delta) = CI(S, \Delta)$, then $\varphi(CI(S, \Delta))|6 - \tilde{\rho}$, where $\tilde{\rho}$ is the Picard number of the minimal resolution of the log canonical cover $\tilde{S}$ and φ is the Euler function.*

Notations and Conventions

In what follows we shall use the following notations.

$A_{n,q}$; A surface singularity which is defined by the automorphism of $\boldsymbol{C}^2$, $\sigma : (x, y) \to (\zeta x, \zeta^q y)$ where $n, q \in \boldsymbol{N}$ and ζ is the primitive n-th root of unity is called the quotient singularity of type $A_{n,q}$.

$(1/n)(w_1, w_2, w_3)$; A 3-dimensional singularity which is defined by the automorphism of $\boldsymbol{C}^3$, $\sigma : (x, y, z) \to (\zeta^{w_1} x, \zeta^{w_2} y, \zeta^{w_3} z)$ where $n, w_i \in \boldsymbol{N}$ for $i = 1, 2, 3$ and ζ is the primitive n-th root of unity is called the quotient singularity of type $(1/n)(w_1, w_2, w_3)$.

Σ_d; Hirzebruch surface of degree d.

$(-n)$-curve; A smooth connected rational curve on a surface with the self intersection number $(-n)$, where $n \in \boldsymbol{N}$.

$\sim$; linear equivalence.

$\sim$num; numerical equivalence.

2 Classification of certain ν_0-log surfaces of type I

In this section we classify the ν_0-log surfaces with Cartier index 2,3,4 and special ν_0-log surface of type K3 with Cartier index 2,3.

Proposition 2.1 *Let (S, Δ) be a ν_0-log surface of abelian type with $CI(S, \Delta) = 2$. Then S is relatively minimal elliptic ruled surface and* SuppΔ *is smooth and Δ is one of the following types.*

(i) $\Delta = (1/2)C$, *where C is a 4-section which is a smooth elliptic curve and $C^2 = 0$.*

(ii) $\Delta = \sum_i^2 (1/2)C_i$, *where C_1 is a 3-section and C_2 is a section. C_i is a smooth elliptic curve and $C_i^2 = 0$ for any i.*

(iii) $\Delta = \sum_i^2 (1/2)C_i$, *where C_i is a 2-section which is a smooth elliptic curve and $C_i^2 = 0$ for any i.*

(iv) $\Delta = \sum_i^3 (1/2)C_i$, *where C_1 is a 2-section which is a smooth elliptic curve, C_i is a section for $i = 2, 3$ and $C_i^2 = 0$ for any i.*

(v) $\Delta = \sum_i^4 (1/2)C_i$, *where C_i is a section and $C_i^2 = 0$ for any i.*

Proposition 2.2 *Let (S, Δ) be a special ν_0-log surface of type K3 with $CI(S, \Delta) = 2$. Then S is a smooth rational surface and* Supp Δ *is smooth. And one of the following holds.*

(i) S is obtained by blowing up $\boldsymbol{P}^2$ or Σ_d $(d = 0, 2, 3, 4)$. $\Delta = \sum_{i=1}^t (1/2)C_i$, where C_1 is a connected smooth curve with genus g, $C_1^2 = 4(g-1)$ and $C_i \simeq \boldsymbol{P}^1$, $C_i^2 = -4$ for $2 \leq i \leq t$. g and t satisfy the following conditions.

$t = g + \rho - 10$, where $\rho := \rho(S)$, $0 \leq g \leq 10$. If $g = 0, 1, 2$, then $1 \leq t \leq 10$. If $g = 3$, then $1 \leq t \leq 7$. If $g = 4, 5, 6$, then $1 \leq t \leq 6$. If $g = 7$ then $1 \leq t \leq 3$. If $g = 8, 9, 10$, then $t = 1, 2$.

(ii) S is obtained by blowing up $\boldsymbol{P}^2$ or Σ_d $(d = 0, 2)$. $\Delta = (1/2)C_1 + (1/2)C_2$, where C_i is a smooth elliptic curve and $C_i^2 = 0$ for $i = 1, 2$. $\rho = 10$.

Proposition 2.3 *Let (S, Δ) be a ν_0-log surface of abelian type with $CI(S, \Delta) = 3$. Then* Supp Δ *is smooth and the following hold.*

(*i*) *S is an elliptic ruled surface with $e = 0$ (For the definition of "e", see [R.Hartshorne], Proposition 2.8). And Δ is one of the following.*

  (*i.a*) $\Delta = (2/3)C$, *where C is a 3-section which is a smooth elliptic curve and $C^2 = 0$.*

  (*i.b*) $\Delta = \sum_{i=1}^{2}(2/3)C_i$, *where C_1 is a 2-section which is a smooth elliptic curve and C_2 is a section. $C_i^2 = 0$ for $i = 1, 2$.*

  (*i.c*) $\Delta = \sum_{i=1}^{3}(2/3)C_i$, *where C_i is a section and $C_i^2 = 0$ for $i = 1, 2, 3$.*

(*ii*) *S is a normal rational surface with $\rho = 4$.* Sing $S = 9A_{3,1}$, *i.e., all of the singular points of S are 9 quotient singular points of type $A_{3,1}$. $\Delta = 0$. The niminal resolution M of S is obtained by blowing up $\boldsymbol{P}^2$ or Σ_d $(d = 0, 2, 3)$.*

Proposition 2.4 *Let (S, Δ) be a special ν_0-log surface of type $K3$ with $CI(S, \Delta) = 3$, then* Sing $S = sA_{3,1}$, *where $s = (1/2)\rho - 1$. The minimal resolution M of S is obtained by blowing up $\boldsymbol{P}^2$, Σ_d $(d = 0, 2, 3, 4, 5, 6)$ and one of the following holds.*

(*i*) $\Delta = 0$, $s = 3$ *and* $\rho = 8$

(*ii*) Supp Δ *is smooth and* Supp $\Delta \cap$ Sing $S = \emptyset$. *Let $\Delta = \sum_{i=1}^{t}(2/3)C_i$ be the irreducible decomposition. Then $C_i \simeq \boldsymbol{P}^1$, $C_i^2 = -6$ for any $2 \leq i \leq t$ and $C_1^2 = 6(g-1)$, $t = (1/2)\rho + g - 4$, where g is the genus of C_1. The range of g is $0 \leq g \leq 5$. If $g = 0$, then $1 \leq t \leq 6$. If $g = 1$, then $1 \leq t \leq 7$. If $g = 2, 3$ then $1 \leq t \leq 4$. If $g = 4$, then $t = 1, 2$. In this case, if t=1, then $S \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$, or Σ_2 and C_1 is a 3-section and if $t = 2$, then M is obtained by blowing up Σ_d $(d = 4, 5, 6)$. If $g = 5$, then $t = 2$, $S \simeq \Sigma_6$, C_1 is a 2-section and C_2 is the negative section.*

Moreover, in each case, if $\rho \neq 2$, then there exists a curve $l_i \subset M$ such that $C_i' \cdot l_j = \delta_{i,j}$ for any i, where C_i' is the strict transform of C_i on M and $\delta_{i,j}$ is the Kronecker's delta.

The last satement is used to determine the singular fibre.

Proposition 2.5 *Let (S, Δ) be a ν_0-log surface of abelian type with $CI(S, \Delta) = 4$, then* Supp Δ *is smooth and one of the follwing holds.*

(*i*) *S is relatively minimal elliptic ruled surface with $e = 0$.*

  (*i.a*) $\Delta = \sum_{i=1}^{2}(3/4)C_{1,i} + (1/2)C_2$, *where $C_{1,i}$, C_2 are sections and $C_{1,i}^2 = C_2^2 = 0$ for $i = 1, 2$.*

  (*i.b*) $\Delta = (3/4)C_1 + (1/2)C_2$, *where C_1 is a 2-section which is a smooth elliptic curve and C_2 is a section and $C_i^2 = 0$ for $i = 1, 2$.*

(*ii*) *S is a normal rational surface with $\rho = 2$ and* $\mathrm{Sing}\ S = 8A_{2,1}$. *The minimal resolution M of S is obtained by blowing up $\boldsymbol{P}^2$, $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ or Σ_2.*

(*ii.a*) $\Delta = \sum_{i=1}^{2}(1/2)C_{2,i}$, *where* $C_{2,i} \simeq \boldsymbol{P}^1$, $C'^2_{2,i} = -2$ *for* $i = 1, 2$. *And* $C_{2,i} \cap \mathrm{Sing\ S} = 4A_{2,1}$ *for* $i = 1, 2$.

(*ii.b*) $\Delta = \sum_{i=1}^{3}(1/2)C_{2,i}$, *where $C_{2,1}$ is a smooth elliptic curve and* $C'^2_{2,1} = 0$, $C_{2,i} \simeq \boldsymbol{P}^1$, $C'^2_{2,i} = -2$ *for* $i = 2, 3$. *And* $C_{2,i} \cap \mathrm{Sing}\ S = 4A_{2,1}$ *for* $i = 1, 2$.

3 Applications to classification of the singular fibres

In this section, we classify the singular fibres in certain cases by using the results in the previous section. In what follows, we assume that $\hat{f} : \hat{X} \longrightarrow \mathcal{D}$ is a projective log minimal degeneration of surfaces with $\kappa = 0$ and that $\hat{\Theta}$ is irreducible.

Theorem 3.1 *Assume that $(\hat{\Theta}, \mathrm{Diff}_{\hat{\Theta}}(0))$ is a ν_0-log surface of abelian type with* $\mathrm{CI}(\hat{\Theta}, \mathrm{Diff}_{\hat{\Theta}}(0)) = 2$, *then there is a minimal projective degeneration $f : X \longrightarrow \mathcal{D}$ which is bimeromorphically equivalent to $\hat{f} : \hat{X} \longrightarrow \mathcal{D}$ (we shrink $\mathcal{D}$ if necessary) such that X is nonsingular, X_t is a abelian or hyperelliptic surface for $t \in \mathcal{D}^*$ and the special fibre $f^*(0)$ is one of the following types.*

(II_α^{ab}) $f^*(0) = 2m\Theta_0 + \sum_{i=1}^{4} m\Theta_i$, *where $m \in \boldsymbol{N}$, Θ_i is an elliptic ruled surface for any i, $\Theta_i|_{\Theta_0}$ is a section of Θ_0 whose self-intersection number 0 for $i \geq 1$. $\Theta_i \cap \Theta_j = \emptyset$ for $i > j \geq 1$.*

(II_β^{ab}) $f^*(0) = 2m\Theta_0 + \sum_{i=1}^{3} m\Theta_i$, *where $m \in \boldsymbol{N}$, Θ_i is an elliptic ruled surface for any i, $\Theta_1|_{\Theta_0}$ is a 2-section of Θ_0 which is a smooth elliptic curve, $\Theta_i|_{\Theta_0}$ is a section of Θ_0 for $i = 2, 3$, $(\Theta_i|_{\Theta_0})^2 = 0$ for $i = 1, 2, 3$. $\Theta_i \cap \Theta_j = \emptyset$ for $i > j \geq 1$.*

(II_γ^{ab}) $f^*(0) = 2m\Theta_0 + \sum_{i=1}^{2} m\Theta_i$, *where $m \in \boldsymbol{N}$, Θ_i is an elliptic ruled surface for $i = 0, 1, 2$, $\Theta_1|_{\Theta_0}$ is a 3-section of Θ_0 which is a smooth elliptic curve, $\Theta_2|_{\Theta_0}$ is a section of Θ_0, $(\Theta_i|_{\Theta_0})^2 = 0$ for $i = 1, 2$. $\Theta_i \cap \Theta_j = \emptyset$ for $i > j \geq 1$.*

(II_δ^{ab}) $f^*(0) = 2m\Theta_0 + \sum_{i=1}^{2} m\Theta_i$, *where $m \in \boldsymbol{N}$, Θ_i is an elliptic ruled surface for $i = 0, 1, 2$, $\Theta_i|_{\Theta_0}$ is a 2-section of Θ_0 which is a smooth elliptic curve with the self-intersection number 0 for $i = 1, 2$. $\Theta_i \cap \Theta_j = \emptyset$ for $i > j \geq 1$.*

(II_ϵ^{ab}) $f^*(0) = 2m\Theta_0 + m\Theta_1$, *where $m \in \boldsymbol{N}$, Θ_i is an elliptic ruled surface for $i = 0, 1$, $\Theta_1|_{\Theta_0}$ is a 4-section of Θ_0 which is a smooth elliptic curve with the self-intersection number 0.*

Theorem 3.2 *Assume that $(\hat{\Theta}, \mathrm{Diff}_{\hat{\Theta}}(0))$ is a special ν_0-log surface of type $K3$ with $\mathrm{CI}(\hat{\Theta}, \mathrm{Diff}_{\hat{\Theta}}(0)) = 2$, then there is a minimal projective degeneration $f : X \longrightarrow \mathcal{D}$ which is bimeromorphically equivalent to $\hat{f} : \hat{X} \longrightarrow \mathcal{D}$ (we shrink $\mathcal{D}$ if necessary) such that X is nonsingular, X_t is a $K3$ surface for $t \in \mathcal{D}^*$ and the special fibre $f^*(0)$ is one of the following types.*

(II^{K3}_{α}-g-t) *$f^*(0) = 2\Theta_0 + \sum_{i=1}^{t} \Theta_i$, where Θ_0 is a smooth rational surface, Θ_1 is a ruled surface and $\Theta_i \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$ or Σ_2 for $2 \leq i \leq t$. $\Theta_1|_{\Theta_0}$ is a smooth connected curve with genus $g := q(\Theta_1)$ whose self-intersection number is $4(g-1)$ and $\Theta_i|_{\Theta_0}$ is a (-4)-curve, i.e., a smooth rational cvurve with the self-intersection number -4 for $i \geq 2$. $\Theta_i \cap \Theta_j = \emptyset$ for $i > j \geq 1$. The relation between t, $\rho(\Theta_0)$ and g is $t = g + \rho(\Theta_0) - 10$. The range of g and t is as follows. $0 \leq g \leq 10$. If $g = 0, 1, 2$, then $1 \leq t \leq 10$. If $g = 3$, then $1 \leq t \leq 7$. If $g = 4, 5, 6$, then $1 \leq t \leq 6$. If $g = 7$ then $1 \leq t \leq 3$. If $g = 8, 9, 10$, then $t = 1, 2$.*

(II^{K3}_{β}) *$f^*(0) = 2\Theta_0 + \sum_{i=1}^{2} \Theta_i$, where Θ_0 is a smooth rational surface with $\rho(\Theta_0) = 10$ and Θ_i is an elliptic ruled surface for $i = 1, 2$. $\Theta_i|_{\Theta_0}$ is a smooth elliptic curve with the self-intersection number 0 for any i. $\Theta_1 \cap \Theta_2 = \emptyset$.*

Theorem 3.3 *Assume that $(\hat{\Theta}, \mathrm{Diff}_{\hat{\Theta}}(0))$ is a ν_0-log surface of abelian type with $\mathrm{CI}(\hat{\Theta}, \mathrm{Diff}_{\hat{\Theta}}(0)) = 3$, then there is a minimal projective degeneration $f : X \longrightarrow \mathcal{D}$ which is bimeromorphically equivalent to $\hat{f} : \hat{X} \longrightarrow \mathcal{D}$ (we shrink $\mathcal{D}$ if necessary) such that X is nonsingular except the case (III^{ab}_{η}-t) below, X_t is a abelian or hyperelliptic surface for $t \in \mathcal{D}^*$ and the special fibre $f^*(0)$ is one of the following types.*

(III^{ab}_{α}) *$f^*(0) = 3m\Theta_0 + \sum_{i=1}^{3}(2m\Theta_{i,1} + m\Theta_{i,2})$, where $m \in \boldsymbol{N}$, Θ_0 and $\Theta_{i,j}$ is an elliptic ruled surface for any i, j. $\Theta_{i,1}|_{\Theta_0}$ is a section of Θ_0 with the self-intersection number 0 and $\Theta_{i,2}|_{\Theta_{i,1}}$ is a section of $\Theta_{i,1}$ for $i = 1, 2, 3$. $\Theta_{i,j} \cap \Theta_{k,l} = \emptyset$ if $i \neq k$ and $\Theta_{i,2} \cap \Theta_0 = \emptyset$ for $i = 1, 2, 3$.*

(III^{ab}_{β}) *$f^*(0) = \sum_{i=1}^{3} m\Theta_i$, where $m \in \boldsymbol{N}$, Θ_i is an elliptic ruled surface for $i = 1, 2, 3$. $\Theta_1 \cap \Theta_2 = \Theta_2 \cap \Theta_3 = \Theta_3 \cap \Theta_1$ is a smooth elliptic curve which is a section on each Θ_i. Θ_i and Θ_j cross normally for $i > j$.*

(III^{ab}_{γ}) *$f^*(0) = 3m\Theta_0 + \sum_{i=1}^{2}(2m\Theta_{i,1} + m\Theta_{i,2})$, where $m \in \boldsymbol{N}$, Θ_0 and $\Theta_{i,j}$ is an elliptic ruled surface for any i, j. $\Theta_{1,1}|_{\Theta_0}$ is a 2-section of Θ_0 which is a smooth elliptic curve with the self-intersection number 0 and $\Theta_{2,1}|_{\Theta_0}$ is a section of Θ_0 with the self-intersection number 0. $\Theta_{i,2}|_{\Theta_{i,1}}$ is a section of $\Theta_{i,1}$ for $i = 1, 2$. $\Theta_{i,j} \cap \Theta_{k,l} = \emptyset$ if $i \neq k$ and $\Theta_{i,2} \cap \Theta_0 = \emptyset$ for $i = 1, 2$.*

(III^{ab}_{δ}) *$f^*(0) = \sum_{i=1}^{2} m\Theta_i$, where $m \in \boldsymbol{N}$. There is a projective birational morphism $\mu : Y \longrightarrow X$ from a smooth 3-fold Y such that $\tilde{f}^*(0) = 3m\tilde{\Theta}_0 +$*

$m\tilde{\Theta}_1 + m\tilde{\Theta}_2$, *where* $\tilde{f} := f \circ \mu$, $\tilde{\Theta}_i := \mu_*^{-1}\Theta_i$ *for* $i = 1, 2$. $\tilde{\Theta}_i$ *is an elliptic ruled surface for* $i = 0, 1, 2$. $\tilde{\Theta}_1|_{\tilde{\Theta}_0}$ *is a 2-section of* $\tilde{\Theta}_0$ *which is a smooth elliptic curve,* $(\tilde{\Theta}_1|_{\tilde{\Theta}_0})^2 = 0$ *and* $\tilde{\Theta}_2|_{\tilde{\Theta}_0}$ *is a section whose self intersection number* 0. $\tilde{\Theta}_1 \cap \tilde{\Theta}_2 = \emptyset$.

(III^{ab}_{ϵ}) $f^*(0) = 3m\Theta_0 + 2m\Theta_1 + m\Theta_2$, *where* $m \in \boldsymbol{N}$, Θ_i *is an elliptic ruled surface for any* i.

$\Theta_1|_{\Theta_0}$ *is a 3-section of* Θ_0 *which is a smooth elliptic curve with the self-intersection number* 0 *and* $\Theta_2|_{\Theta_1}$ *is a section of* Θ_1 *with the self-intersection number* 0. $\Theta_0 \cap \Theta_2 = \emptyset$.

(III^{ab}_{ζ}) $f^*(0) = m\Theta$, *where* $m \in \boldsymbol{N}$. *There is a projective birational morphism* $\mu : Y \to X$ *from a smooth 3-fold* Y *such that* $\tilde{f}^*(0) = 3m\tilde{\Theta}_0 + m\tilde{\Theta}_1$, *where* $\tilde{f} := f \circ \mu$, $\tilde{\Theta}_1 := \mu_*^{-1}\Theta_1$. $\tilde{\Theta}_i$ *is an elliptic ruled surface for* $i = 0, 1$. $\tilde{\Theta}_1|_{\tilde{\Theta}_0}$ *is a 3-section of* $\tilde{\Theta}_0$ *which is a smooth elliptic curve and* $(\tilde{\Theta}_1|_{\tilde{\Theta}_0})^2 = 0$.

(III^{ab}_{η}-t) $f^*(0) = 3\Theta_0 + \sum_{i=1}^{t}\Theta_i$, *where* Θ_0 *is a normal rational surface and* $\Theta_i \simeq \boldsymbol{P}^2$ *for* $i \geq 1$. $\operatorname{Sing}\Theta_0 = sA_{3,1}$,*where* $s := 9 - t$. $\Theta_i|_{\Theta_0}$ *is a* (-3)*-curve for* $i \geq 1$. $\operatorname{Sing} X$ *equals to* $\operatorname{Sing}\Theta_0$ *set theoretically and each singular point of* X *is a quotient singularity of type* $(1/3)(1,2,2)$. *Moreover, if* X_t *is an abeian surface for* $t \in \mathcal{D}^*$, *then* $t = 0$ *or* 9.

Theorem 3.4 *Assume that* $(\hat{\Theta}, \mathrm{Diff}_{\hat{\Theta}}(0))$ *is a special* ν_0*-log surface of type* $K3$ *with* $\mathrm{CI}(\hat{\Theta}, \mathrm{Diff}_{\hat{\Theta}}(0)) = 3$, *then there is a minimal projective degeneration* $f : X \to \mathcal{D}$ *which is bimeromorphically equivalent to* $\hat{f} : \hat{X} \to \mathcal{D}$ *(we shrink* $\mathcal{D}$ *if necessary) such that* X *is nonsingular except the case* (III^{K3}_{β}-g-t-s) *below,* X_t *is a* $K3$ *surface for* $t \in \mathcal{D}^*$ *and the special fibre* $f^*(0)$ *is one of the following types.*

(III^{K3}_{α}-g-t-s) $f^*(0) = 3\Theta_0 + \sum_{i=1}^{t}(2\Theta_{i,1} + \Theta_{i,2}) + \sum_{j=1}^{s}\Theta_j$, *where* Θ_0 *is a smooth rational surface,* $\Theta_{1,j}$ *is a ruled surface for* $j = 1, 2$, $\Theta_{i,1} \simeq \Sigma_4$, $\Theta_{i,2} \simeq \Sigma_2$ *or* $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ *for* $i \geq 2$ *and* $\Theta_j \simeq \boldsymbol{P}^2$ *for any* j. $\Theta_{1,1}|_{\Theta_0}$ *is a smooth connected curve with the genus* $g := q(\Theta_{1,1})$ *whose self intersection number is* $6(g-1)$, $\Theta_{i,1}|_{\Theta_0}$ *is a* (-6)*-curve for* $i \geq 2$ *and* $\Theta_j|_{\Theta_0}$ *is a* (-3)*-curve for any* j. $\Theta_{i,2}|_{\Theta_{i,1}}$ *is a section of* $\Theta_{i,1}$ *for* $1 \leq i \leq t$. $\Theta_{i,j} \cap \Theta_{k,l} = \emptyset$ *if* $i \neq k$ $\Theta_{i,j} \cap \Theta_k = \emptyset$ *for any* i, j, k, $\Theta_i \cap \Theta_j = \emptyset$ *for* $i > j$ *and* $\Theta_{i,2} \cap \Theta_0 = \emptyset$ *for any* i. $\rho(\Theta_0) = 3t - 3g + 11$, $s = t - g + 3$ *and the range of* g *is* $0 \leq g \leq 5$. *The range of* t *is as follows. If* $g = 0$, *then* $0 \leq t \leq 6$, *if* $g = 1$, *then* $1 \leq t \leq 7$, *if* $g = 2, 3$, *then* $1 \leq t \leq 4$, *if* $g = 4$, *then* $t = 1, 2$ *and if* $g = 5$, *then* $t = 2$.

(III^{K3}_{β}-g-t-s) $f^*(0) = 3\Theta_0 + \sum_{i=1}^{t}(2\Theta_{i,1} + \Theta_{i,2})$, *where* Θ_0 *is a normal rational surface,* $\Theta_{1,j}$ *is a ruled surface for* $j = 1, 2$, $\Theta_{i,1} \simeq \Sigma_4$, $\Theta_{i,2} \simeq \Sigma_2$ *or* $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ *for* $i \geq 2$. *All singular points of* Θ_0 *are disjoint from any other* *component of the singular fibre.* $\Theta_{1,1}|_{\Theta_0}$ *is a smooth connected curve with the genus* $g := q(\Theta_{1,1})$ *whose self intersection number is* $6(g-1)$, $\Theta_{i,1}|_{\Theta_0}$ *is a* (-6)*-curve for* $i \geq 2$. $\Theta_{i,2}|_{\Theta_{i,1}}$ *is a section of* $\Theta_{i,1}$ *for* $1 \leq i \leq t$. $\Theta_{i,2} \cap \Theta_0 = \emptyset$ *for any* i *and* $\Theta_{i,j} \cap \Theta_{k,l} = \emptyset$ *if* $i \neq k$. $\mathrm{Sing}\, \Theta_0 = sA_{3,1}$, *where* $s := t - g + 3$. $\mathrm{Sing}\, X = \mathrm{Sing}\, \Theta_0$ *set theoretically and each singular point of* X *is a quotient singularity of type* $(1/3)(1,2,2)$. $\rho(\Theta_0) = 2t - 2g + 8$ *and the range of* g *and* t *is the same as in the case* (i) *above.*

(III^{K3}_γ) $f^*(0) = \Theta$, *where* Θ *is irreducible. There is a projective birational morphism* $\mu : Y \longrightarrow X$ *from a smooth 3-fold* Y *such that* $\tilde{f}^*(0) = 3\tilde{\Theta}_0 + \tilde{\Theta}_1$, *where* $\tilde{f} := f \circ \mu$, $\tilde{\Theta}_1 := \mu_*^{-1}\Theta$, $\tilde{\Theta}_0$ *is exceptional for* μ, $\tilde{\Theta}_0 \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$ *or* Σ_2, $\tilde{\Theta}_1$ *is a ruled surface with irregularity* 4, $\tilde{\Theta}_1|_{\tilde{\Theta}_0}$ *is a 3-section which is a smooth curve with the genus* 4 *whose self intersection number* 18.

(III^{K3}_δ) $f^*(0) = \Theta_1 + \Theta_2$, *where* Θ_i *is irreducible for* $i = 1,2$. *There is a projective birational morphism* $\mu : Y \longrightarrow X$ *from a smooth 3-fold* Y *such that* $\tilde{f}^*(0) = 3\tilde{\Theta}_0 + \tilde{\Theta}_1 + \tilde{\Theta}_2$, *where* $\tilde{f} := f \circ \mu$, $\tilde{\Theta}_i := \mu_*^{-1}\Theta_i$*for* $i = 1,2$, $\tilde{\Theta}_0$ *is exceptional for* μ, $\tilde{\Theta}_0 \simeq \Sigma_6$, $\tilde{\Theta}_1$ *is a ruled surface with irregularity* 5, $\tilde{\Theta}_2 \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$ *or* Σ_2, $\tilde{\Theta}_1|_{\tilde{\Theta}_0}$ *is a 3-section which is a smooth curve with the genus* 5 *whose self intersection number* 24, $\tilde{\Theta}_2|_{\tilde{\Theta}_0}$ *is a negative section of* $\tilde{\Theta}_0$. $\tilde{\Theta}_1 \cap \tilde{\Theta}_2 = \emptyset$.

Theorem 3.5 *Assume that* $(\hat{\Theta}, \mathrm{Diff}_{\hat{\Theta}}(0))$ *is a* ν_0*-log surface of abelian type with* $\mathrm{CI}(\hat{\Theta}, \mathrm{Diff}_{\hat{\Theta}}(0)) = 4$, *then there is a minimal projective degeneration* $f : X \longrightarrow \mathcal{D}$ *which is bimeromorphically equivalent to* $\hat{f} : \hat{X} \longrightarrow \mathcal{D}$ *(we shrink* $\mathcal{D}$ *if necessary) such that* X *is nonsingular except the case* (IV^{ab}_ζ) *and* (IV^{ab}_η) *below,* X_t *is a abelian or hyperelliptic surface for* $t \in \mathcal{D}^*$ *and the special fibre* $f^*(0)$ *is one of the following types.*

(IV^{ab}_α) $f^*(0) = m\Theta_1 + m\Theta_2$, *where* $m \in \boldsymbol{N}$, Θ_i *is an elliptic ruled surface for* $i = 1,2$. $\Theta_1|_{\Theta_2} = 2C$ *where* C *is a section of* Θ_2.

(IV^{ab}_β) $f^*(0) = 4m\Theta_0 + \sum_{i=1}^2(3m\Theta_{i,1} + 2m\Theta_{i,2} + m\Theta_{i,3}) + 2m\Theta_3$, *where* Θ_0, $\Theta_{i,j}$, Θ_3 *are elliptic ruled surface.* $\Theta_{i,1}|_{\Theta_0}$ $(i = 1,2)$, $\Theta_3|_{\Theta_0}$ *are sections of* Θ_0 *whose self intersection number* 0. $\Theta_{i,3}|_{\Theta_{i,2}}$ *and* $\Theta_{i,2}|_{\Theta_{i,1}}$ *are sections of* $\Theta_{i,2}$ *and* $\Theta_{i,1}$ *respectively whose self-intersection number* 0. $\Theta_{i,j} \cap \Theta_{k,l} = \emptyset$ *if* $i \neq k$, $\Theta_{i,3} \cap \Theta_{i,1} = \emptyset$ *for* $i = 1,2$ *and* $\Theta_3 \cap \Theta_{i,j} = \emptyset$ *for any* i, j.

(IV^{ab}_δ) $f^*(0) = m\Theta$, *where* $\boldsymbol{N}$, Θ *is irreducible. there is a projective birational morphism* $\mu : Y \longrightarrow X$ *from a smooth 3-fold* Y *such that* $\tilde{f}^*(0) = 4m\tilde{\Theta}_0 + m\tilde{\Theta}_1 + m\tilde{\Theta}_2$, *where* $\tilde{f} := f \circ \mu$, Θ_i *is an elliptic ruled surface,* $\tilde{\Theta}_1 = \mu_*^{-1}\Theta$. $\tilde{\Theta}_1|_{\tilde{\Theta}_0}$ *is a 2-section of* $\tilde{\Theta}_0$ *which is a smooth elliptic curve,* $\tilde{\Theta}_2|_{\tilde{\Theta}_0}$ *is a section of* $\tilde{\Theta}_0$ *and* $(\tilde{\Theta}_i|_{\tilde{\Theta}_0})^2 = 0$ *for* $i = 1,2$. $\tilde{\Theta}_1 \cap \tilde{\Theta}_2 = \emptyset$.

(IV_{ϵ}^{ab}) $f^*(0) = 4m\Theta_0 + 3m\Theta_1 + 2m\Theta_2 + m\Theta_3 + 2m\Theta_4$, *where* $m \in \boldsymbol{N}$, Θ_i *is an elliptic ruled surface for any* i, $\Theta_1|_{\Theta_0}$ *is a 2-section which is a smooth elliptic curve,* $\Theta_4|_{\Theta_0}$ *is a section of* Θ_0, $(\Theta_i|_{\Theta_0})^2 = 0$ *for* $i = 1, 4$. $\Theta_2|_{\Theta_1}$ *and* $\Theta_3|_{\Theta_2}$ *are sections of* Θ_1, Θ_1 *respectively.* $\Theta_i \cap \Theta_j = \emptyset$ *for* $(i,j) = (1,4), (2,4), (2,0), (3,1), (3,4), (3,0)$.

(IV_{ζ}^{ab}) $f^*(0) = 4\Theta_0 + \sum_{i=1}^{3} 2\Theta_i + \sum_{i=2}^{3}\sum_{j_i=1}^{t_i} \Theta_{i,j_i}$, *where* Θ_i *is a normal rational surface for* $i = 0, 2, 3$, Θ_1 *is an elliptic ruled surface and* $\Theta_{i,j_i} \simeq \Sigma_2$. $t_i = 0$ *or* 2 *or* 4 *for* $i = 2, 3$ *and* $s := 8 - \sum_{i=2}^{3} t_i$. $\Theta_1|_{\Theta_0}$ *is a smooth elliptic curve whose self intersection number 0,* $\Theta_i|_{\Theta_0} \simeq \boldsymbol{P}^1$ *for* $i = 2, 3$, $\Theta_{i,j_i}|_{\Theta_0}$ *is a* (-2)*-curve for any* (i, j_i), $\Theta_{i,j_i}|_{\Theta_i}$ *is a* (0)*-curve.* $\Theta_i \cap \Theta_j = \emptyset$ *for* $i > j$, $\Theta_{i,j_i} \cap \Theta_k = \emptyset$ *if* $i \neq k$ *and* Θ_{i,j_i}*'s are disjoint from each other.* Sing $\Theta_0 = \{P_{1,j_i}^{(i)} \in \Theta_i; 0 \leq j_i \leq 8 - t_i\ (i = 2, 3)\}$ *and any* $P_{1,j_i}^{(i)} \in \Theta_0$ *is of type* $A_{2,1}$. Sing $\Theta_i = \{P_{1,j_i}^{(i)},\ P_{2,j_i}^{(i)}; 0 \leq j_i \leq 8 - t_i\ (i = 2, 3)\}$ *and any* $P_{i,j_i}^{(i)} \in \Theta_i$ *is of type* $A_{2,1}$ *for* $i = 2, 3$. Sing $X = \{P_{1,j_i}^{(i)},\ P_{2,j_i}^{(i)}; 0 \leq j_i \leq 8 - t_i\ (i = 2, 3)\}$ *and any* $P_{i,j_i}^{(i)} \in X$ *is quotient singularity of type* $(1/2)(1,1,1)$. *Moreover, if* X_t *is an abelian surface for* $t \in \mathcal{D}^*$, *then* $(t_2, t_3) = (0, 0)$, *or* $(4, 4)$.

(IV_{η}^{ab}) $f^*(0) = 4\Theta_0 + \sum_{i=1}^{2} 2\Theta_i + \sum_{i=1}^{2}\sum_{j_i=1}^{t_i} \Theta_{i,j_i}$, *where* Θ_i *is a normal rational surface for* $i = 0, 1, 2$, $\Theta_{i,j_i} \simeq \Sigma_2$. $t_i = 0$ *or* 2 *or* 4 *for* $i = 1, 2$ *and* $s := 8 - \sum_{i=2}^{3} t_i$. $\Theta_i|_{\Theta_0} \simeq \boldsymbol{P}^1$ *for* $i = 1, 2$, $\Theta_{i,j_i}|_{\Theta_0}$ *is a* (-2)*-curve for any* (i, j_i), $\Theta_{i,j_i}|_{\Theta_i}$ *is a* (0)*-curve.* $\Theta_1 \cap \Theta_2$. $\Theta_{i,j_i} \cap \Theta_k = \emptyset$ *if* $i \neq k$ *and* Θ_{i,j_i}*'s are disjoint from each other.* Sing $\Theta_0 = \{P_{1,j_i}^{(i)} \in \Theta_i; 0 \leq j_i \leq 8 - t_i\ (i = 1, 2)\}$ *and any* $P_{1,j_i}^{(i)} \in \Theta_0$ *is of type* $A_{2,1}$. Sing $\Theta_i = \{P_{1,j_i}^{(i)},\ P_{2,j_i}^{(i)}; 0 \leq j_i \leq 8 - t_i\ (i = 1, 2)\}$ *and any* $P_{i,j_i}^{(i)} \in \Theta_i$ *is of type* $A_{2,1}$ *for* $i = 1, 2$. Sing $X = \{P_{1,j_i}^{(i)},\ P_{2,j_i}^{(i)}; 0 \leq j_i \leq 8 - t_i\ (i = 1, 2)\}$ *and any* $P_{i,j_i}^{(i)} \in X$ *is quotient singularity of type* $(1/2)(1,1,1)$. *Moreover, if* X_t *is an abelian surface for* $t \in \mathcal{D}^*$, *then* $(t_1, t_2) = (0, 0)$, *or* $(4, 4)$.

4 Idea of the proof

In this section we give the outline of the proof of Proposition 2.4. Theorem 3.4 can be deduced from this proposition as in the same way in [K.Ohno]. Let notations be as above. Firstly, as for the singularities of S and the boundary, we have the following lemma.

Lemma 4.1 *Let* (S, Δ) *be a special* ν_0*-log surface of type* $K3$ *with* $\mathrm{CI}(S, \Delta) = 3$, *then following (i), (ii), (iii) hold.*

(i) All singular poits of S *are of type* $A_{3,1}$.

(ii) Sing $S \cap \mathrm{Supp}\ \Delta = \emptyset$.

(*iii*) SuppΔ *is smooth.*

The above lemma is checked by considering actions of $Gal\,(\tilde{S}/S)$ around fixed points. Δ is written as $\Delta = (2/3)C$, where C is a reduced smooth curve.

Lemma 4.2 *Let (S, Δ) be as above and let $\mu : M \to S$ be the minimal resolution of S and s be the number of singular points of S. Then the following formulae hold.*

(*i*) $s = (1/2)\rho - 1$, *where* $\rho := \rho(S)$.

(*ii*) $K_M^2 = 11 - (3/2)\rho$.

(*iii*) $K_M \cdot C = 2\rho - 16$ *and* $C^2 = 24 - 3\rho$, *hence* $K_M \cdot C + C^2 = 8 - \rho$.

Proof. Put $U := S \setminus (\mathrm{Supp}\ \Delta \cup \mathrm{Sing}\ S)$. Since $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is étale, we have

$$\chi_{\mathrm{top}}(\tilde{S}) - \chi_{\mathrm{top}}(\pi^{-1}C) - s = 3(\chi_{\mathrm{top}}(M) - \chi_{\mathrm{top}}(C') - 2s), \tag{1}$$

where $C' := \mu_*^{-1}C$. And from the definition, we have

$$K_M + \frac{2}{3}C' + \frac{1}{3}E \sim \mathrm{num}\ 0, \tag{2}$$

where $E := \mu^{-1}(\mathrm{Sing}\ S)$. From (1) and (2), we obtain the desired result. ∎

Let $\tau : M \to N$ be a birational morphism form M to a relatively minimal model N,

$$M_n := M \xrightarrow{\tau_n} M_{n-1} \xrightarrow{\tau_{n-1}} \dots \xrightarrow{\tau_1} M_0 := N \tag{3}$$

be the decomposition of τ to the sequence of contractions of (-1)-curves. Let $\sigma_i : M \to M_i$ be the induced morphism and put $C^{(i)} := \sigma_{i*}C'$, $E^{(i)} := \sigma_{i*}E$, $m_i := \mathrm{mult}_{p_i}C^{(i)}$, $m_i' := \mathrm{mult}_{p_i}E^{(i)}$, where p_i is the center of the blow-up τ_{i+1}. Let F_i be the exceptional divisor of τ_i. Case (α). If F_{i+1} is not contained in $C^{(i+1)} \cup E^{(i+1)}$, then $(m_i, m_i') = (0,3)$ or $(1,1)$. In the case $(m_i, m_i') = (0,3)$ (resp., $(1,1)$), we call p_i is of type $(0,3)$ (resp., of type $(1,1)$) and we call τ_{i+1} $(\alpha.1)$-blow up (resp., $(\alpha.2)$-blow up). Case (β). If $F_{i+1} \subseteq C^{(i+1)}$, then $(m_i, m_i') = (0,5)$, $(1,3)$ or $(2,1)$). In the case $(m_i, m_i') = (0,5)$ (resp., $(1,3)$, resp., $(2,1)$), we call p_i is of type $(0,5)$ (resp., of type $(1,3)$, resp., $(2,1)$) and we call τ_{i+1} $(\beta.1)$-blow up (resp., $(\beta.2)$-blow up, resp., $(\beta.3)$-blow up). Case (δ). If $F_{i+1} \subseteq E^{(i+1)}$, then $(m_i, m_i') = (0,4)$, $(1,2)$ or $(2,0)$). In the case $(m_i, m_i') = (0,4)$ (resp., $(1,2)$, resp., $(2,0)$), we call p_i is of type $(0,4)$ (resp., of type $(1,2)$, resp., $(2,0)$) and we call τ_{i+1} $(\delta.1)$-blow up (resp., $(\delta.2)$-blow up, resp., $(\delta.3)$-blow up).

The following lemma is easily derived.

Lemma 4.3 *Define* α_1 *as the number of the* $(\alpha.1)$*-blowing up which appears in the sequence* (3) *and so on. And put* $\Phi_1 := K_M \cdot C - K_N \cdot C^{(0)}$, $\Phi_2 := (1/2)(K_N \cdot C^{(0)} + C^{(0)2} - K_M \cdot C - C^2)$ *and* $\Phi_3 := K_M \cdot E - K_N \cdot E^{(0)}$. *Then the following formulae hold.*

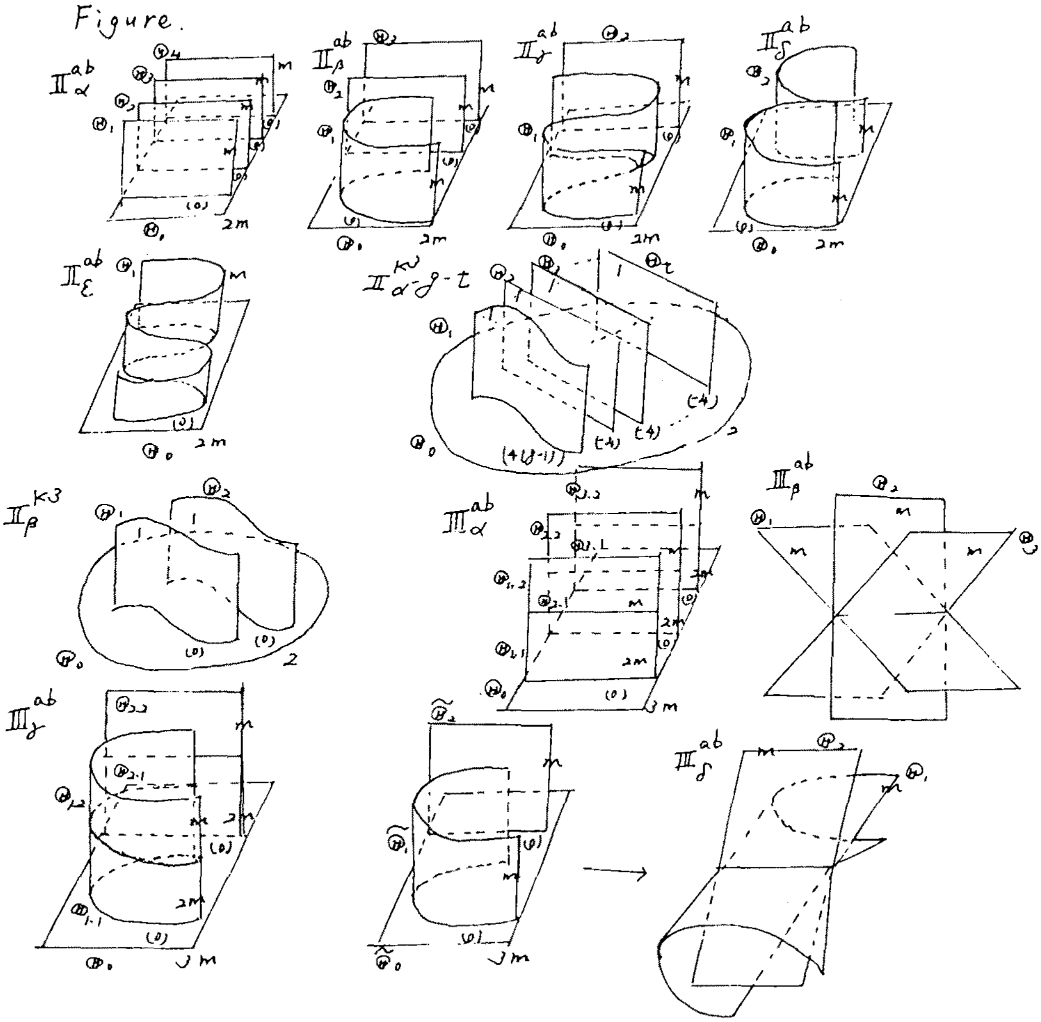
(1) $\alpha_2 + \beta_3 + \gamma_2 = \beta_1 - 2\gamma_3 + \Phi_1$.

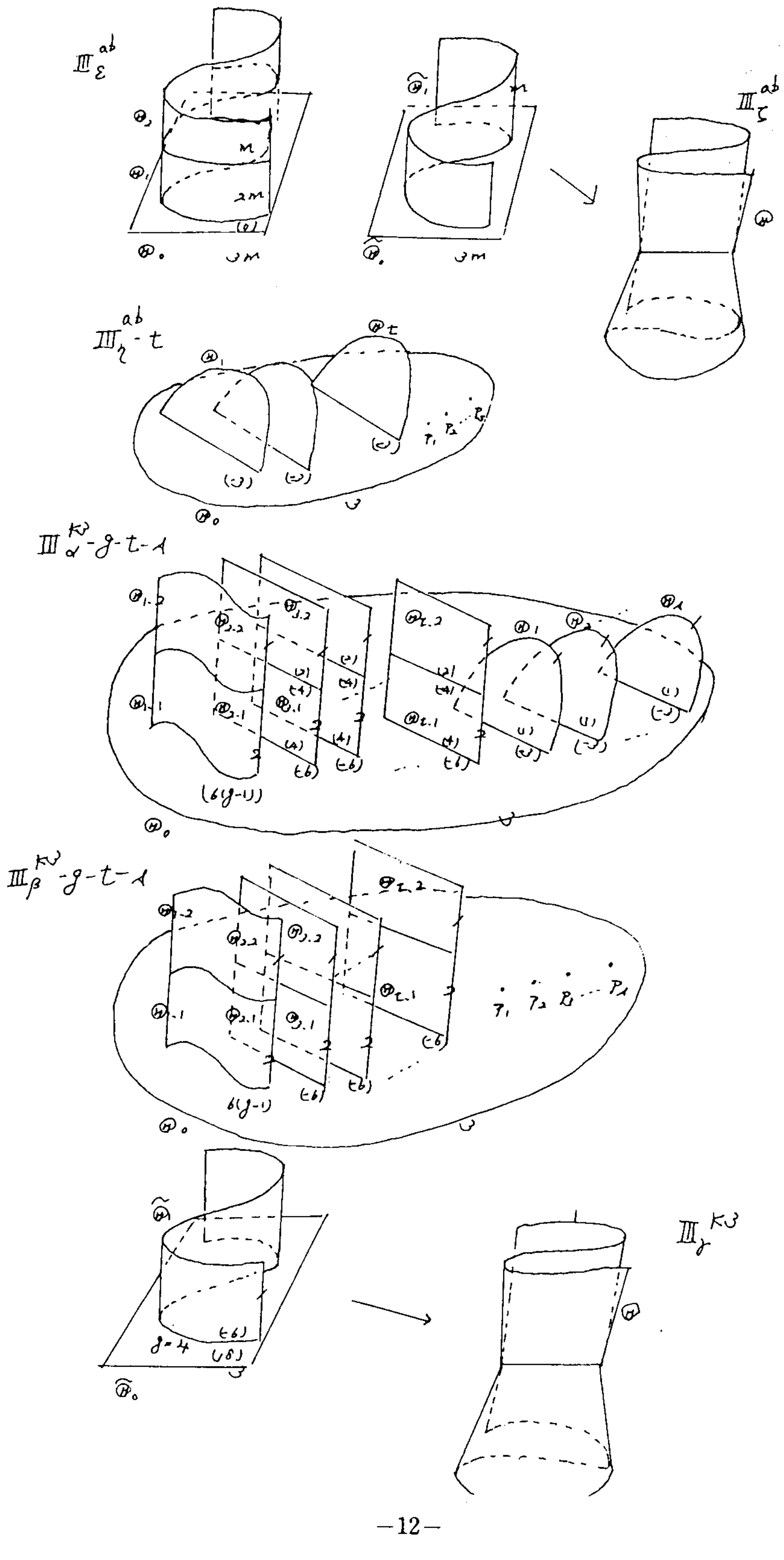
(2) $\beta_1 + \gamma_3 = \Phi_2$.

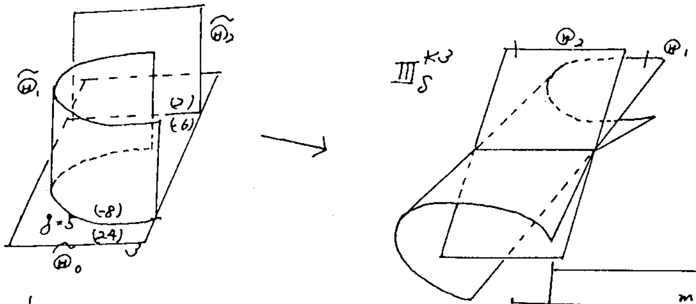
(3) $\alpha_1 + \beta_2 + \gamma_1 = (1/3)(\Phi_3 - \Phi_1) - 2\beta_1 + \gamma_3$.

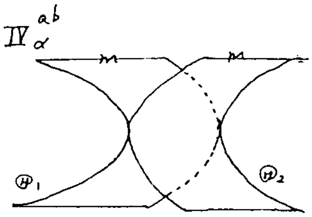
If $N \simeq \boldsymbol{P}^2$ for example, $(\deg C^{(0)}, \deg E^{(0)}) = (0,9), (1,7), (2,5), (3,3)$ or $(4,1)$. Proposition 2.4 is derived from Lemma 4.2 and Lemma 4.3 by checking case by case.

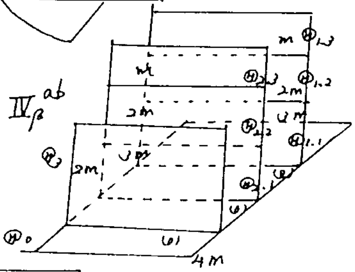
Figure.

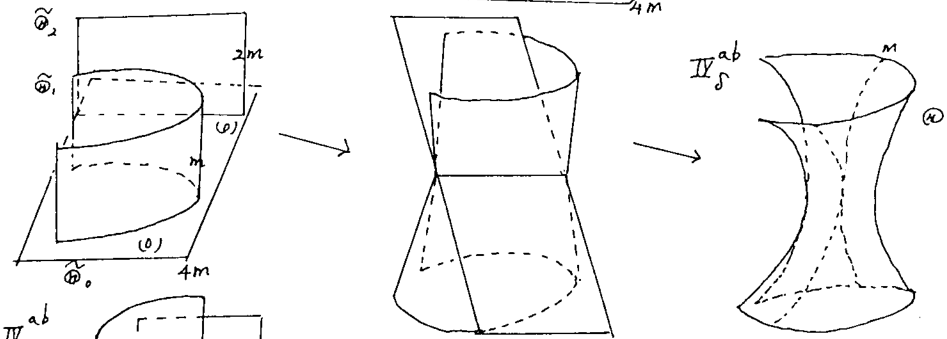


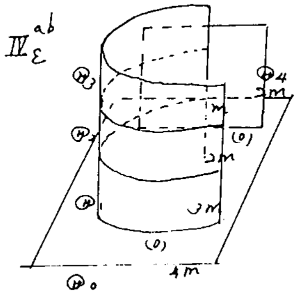

III_{ε}^{ab}
Θ_2
Θ_1
m
2m
Θ_0
3m
$\tilde{\Theta}_1$
m
$\tilde{\Theta}_0$
3m
III_{ζ}^{ab}
III_{η}^{ab} - t
Θ_1
Θ_t
p_1
p_2
Θ_0
III_{α}^{K3} - g - t - λ
$\Theta_{1.2}$
$\Theta_{2.2}$
$\Theta_{3.2}$
$\Theta_{t.2}$
$\Theta_{1.1}$
$\Theta_{2.1}$
$\Theta_{3.1}$
$\Theta_{t.1}$
Θ_1
Θ_2
Θ_λ
(6(g-1))
Θ_0
III_{β}^{K3} - g - t - λ
$\Theta_{1.2}$
$\Theta_{2.2}$
$\Theta_{3.2}$
$\Theta_{t.2}$
$\Theta_{1.1}$
$\Theta_{2.1}$
$\Theta_{3.1}$
$\Theta_{t.1}$
p_1
p_2
p_3
p_λ
6(g-1)
Θ_0
$\tilde{\Theta}$
g=4
(-6)
(8)
$\tilde{\Theta}_0$
III_{γ}^{K3}

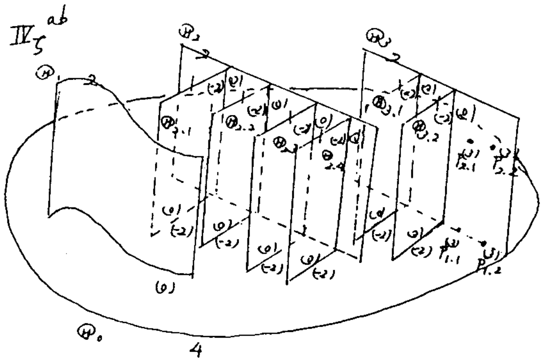




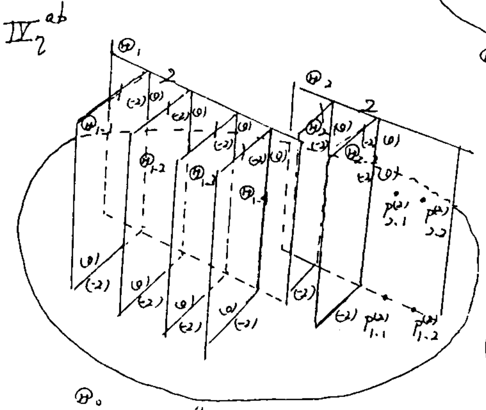








(Case $(t_2, t_3) = (4, 2)$)



(Case $(t_1, t_2) = (4, 2)$)

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